

# Entropy production and fluctuation relations for a KPZ interface

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**Abstract.** We study entropy production and fluctuation relations in the restricted solid-on-solid growth model, which is a microscopic realization of the KPZ equation. Solving the one dimensional model exactly on a particular line of the phase diagram we demonstrate that entropy production quantifies the distance from equilibrium. Moreover, as an example of a physically relevant current different from the entropy, we study the symmetry of the large deviation function associated with the interface height. In a special case of a system of length  $L = 4$  we find that the probability distribution of the variation of height has a symmetric large deviation function, displaying a symmetry different from the Gallavotti-Cohen symmetry.

## 1. Introduction

The theoretical understanding of out of equilibrium systems is one of the main goals in statistical physics. Nonequilibrium phenomena display very rich behavior, with several features not observed in equilibrium, as for example phase transitions in one-dimensional systems with short-range interactions. Recently great theoretical progress has been made concerning systems in contact with two reservoirs [1–5]. Nevertheless, for general nonequilibrium systems very little is known.

For some continuous time Markov jump processes an (average) entropy production can be defined [6, 7]. This quantity is zero if detailed balance is fulfilled, which corresponds to an equilibrium stationary state, and it is positive in the case of a nonequilibrium stationary state. Therefore, a positive entropy production in the stationary state is a signature of nonequilibrium. In spite of the generality of this statement the relation between entropy production and nonequilibrium stationary states is still poorly understood [8]. Recent studies in this direction concern the relation between entropy production and nonequilibrium phase transitions, where the entropy

production as a function of the control parameter has been observed to peak near the critical point [9–11].

In this paper we will consider a microscopic surface growth model in the Kardar-Parisi-Zhang [12] (KPZ) universality class. The KPZ equation reads

$$\partial_t h(x, t) = v + \nu \nabla^2 h(x, t) + (\lambda/2)(\nabla h(x, t))^2 + \eta(x, t), \quad (1)$$

where  $x$  denotes the position on a  $d$ -dimensional interface (we will be dealing only with the  $d = 1$  case),  $h(x, t)$  represents the height of the interface and  $\eta(x, t)$  is a Gaussian white noise. The Laplacian term is related to surface tension,  $v$  is the velocity of the interface at zero slope and the non-linear term is the lowest order term that breaks the up-down symmetry ( $h(x, t) \rightarrow -h(x, t)$ ) [13, 14]. Assuming that a nonequilibrium stationary state of a KPZ interface is characterized by  $v$  and  $\lambda$ , we study how the entropy production varies with these parameters. We shall argue that this is a reasonable assumption and it allows us to investigate how entropy production characterizes a nonequilibrium stationary state. We will work with the restricted solid on solid (RSOS) growth model, because this model has a rich phase diagram and we can perform analytical calculations (exact and approximative).

The positivity of the average entropy production is a statement analogous to the second law of thermodynamics. More generally, it is possible to associate a fluctuating entropy with a single stochastic path [7], and the probability distribution of this fluctuating entropy is constrained through the fluctuation relation [15–24], which is a stronger statement than the positivity of the average entropy production. The fluctuation relation implies the existence of a symmetry of the large deviation function associated with the probability distribution of entropy which is known as the Gallavotti-Cohen (GC) symmetry [17, 19, 23].

In general, unlike the entropy production, other time-integrated currents are not expected to have symmetric large deviation function [19]. In a growing interface, the physically relevant time-integrated current is the variation of the interface height. The main focus of this work is to analyze the symmetry of the large deviation function associated with the variation of height using the RSOS model. We find that while indeed no symmetry of the large deviation function exists in the general case, the large deviation function related to the variation of height is symmetric in the case of a four-site system. This is in spite of the fact that entropy production and height are not proportional to each other in the long time limit. Unlike the GC symmetry which results from a certain relation between the rate of each microscopic trajectory and its time reversed one, this symmetry is a result of a similar relation between the rates associated with certain groups of trajectories.

The paper is organized as follows. In Sec. 2 we define entropy and briefly discuss its relation with nonequilibrium stationary states. In Sec. 3 we define the RSOS model and calculate the average entropy production analytically, analyzing how it depends

on the interface velocity and  $\lambda$ . The fluctuation relation and the GC symmetry are introduced in Sec. 4. In Sec. 5 we investigate fluctuations of the interface height. We show that if a second time-integrated current is considered a fluctuation relation for the joint probability distribution of two currents, with one of the currents being the height, can be found and we also show that for the four-site system the large deviation function related to height is symmetric, displaying a symmetry different from GC symmetry. We conclude in Sec. 6.

## 2. Entropy production in nonequilibrium stationary states

In the following discussion we consider continuous time Markov processes that obey the master equation

$$\frac{d}{dt}P(C, t) = \sum_{C' \neq C} [w_{C' \rightarrow C}P(C', t) - w_{C \rightarrow C'}P(C, t)], \quad (2)$$

where  $P(C, t)$  is the probability of being in state  $C$  at time  $t$  and  $w_{C \rightarrow C'}$  is the transition rate from state  $C$  to  $C'$ . For such stochastic processes three different fluctuating entropies can be defined, namely [7]

- (a) the *configurational entropy* of the system  $S_{sys}(t) = -\ln P(C, t)$ ,
- (b) the *entropy of the medium* (=environment)  $S_m(t)$  which changes instantaneously by  $\ln(w_{C \rightarrow C'}/w_{C' \rightarrow C})$  during a transition from  $C$  to  $C'$ .
- (c) and the *total entropy*  $S_{tot}(t) = S_{sys}(t) + S_m(t)$ .

Note that the definition of the environmental entropy requires the rates to satisfy the condition that if  $w_{C \rightarrow C'} > 0$  then the reverse rate  $w_{C' \rightarrow C}$  has to be positive as well.

A stochastic path is a sequence of transitions

$$C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \dots C_N,$$

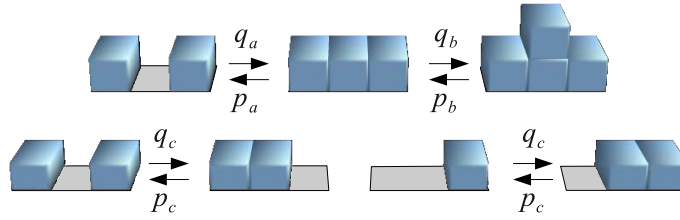
taking place at random instances of time  $T_0 \leq t_1 < t_2 < \dots < t_N \leq T$  according to the specific rates during a fixe time interval  $\Delta T = T - T_0$ . Such a path of transitions changes the aforementioned entropies by

$$\Delta S_{sys} = \ln P(C_0, t_0) - \ln P(C_N, t_N), \quad (3)$$

$$\Delta S_m = \sum_{i=1}^N \ln \frac{w_{C_{i-1} \rightarrow C_i}}{w_{C_i \rightarrow C_{i-1}}}, \quad (4)$$

and  $\Delta S_{tot} = \Delta S_{sys} + \Delta S_m$ . While these entropies vary discontinuously for a particular stochastic path, their *expectation values* vary smoothly according to

$$\dot{S}_{sys}(t) := \frac{d}{dt} \langle S_{sys}(t) \rangle = \sum_{C, C'} \ln \frac{P(C, t)}{P(C', t)} P(C, t) w_{C \rightarrow C'}, \quad (5)$$



**Figure 1.** Deposition and evaporation rates for the RSOS model.

$$\dot{s}_m(t) := \frac{d}{dt} \langle S_m(t) \rangle = \sum_{C, C'} \ln \frac{w_{C \rightarrow C'}}{w_{C' \rightarrow C}} P(C, t) w_{C \rightarrow C'}, \quad (6)$$

$$\dot{s}_{tot}(t) := \frac{d}{dt} \langle S_m(t) \rangle = \sum_{C, C'} \ln \frac{P(C, t) w_{C \rightarrow C'}}{P(C', t) w_{C' \rightarrow C}} P(C, t) w_{C \rightarrow C'}, \quad (7)$$

where  $\langle \dots \rangle$  denotes the ensemble average. Since the last equation can be rewritten as

$$\dot{s}_{tot}(t) = \frac{1}{2} \sum_{C, C'} \left( \ln[P(C, t) w_{C \rightarrow C'}] - \ln[P(C', t) w_{C' \rightarrow C}] \right) \left( P(C, t) w_{C \rightarrow C'} - P(C', t) w_{C' \rightarrow C} \right) \quad (8)$$

and both brackets have always the same sign, the average total entropy can only increase and will stay constant if detailed balance is satisfied, in accordance with the second law of thermodynamics. Moreover, if the system is in a non-equilibrium stationary state, the average internal entropy  $\langle S_{sys}(t) \rangle = - \sum_C P(C) \ln P(C)$  is constant so that  $\dot{s}_{tot} = \dot{s}_m$ . Since the stationary master equation

$$0 = \sum_{C'} w_{C' \rightarrow C} P(C') - \sum_{C'} w_{C \rightarrow C'} P(C) \quad (9)$$

is invariant under the replacement  $w_{C \rightarrow C'} \rightarrow w_{C \rightarrow C'} + A(C, C')/P(C)$  with an arbitrary symmetric function  $A(C, C') = A(C', C)$  while Eq. (8) is not invariant under this operation, the same stationary state may be generated by different dynamical rules with a different entropy production rate (see [8]).

### 3. Entropy production in a solid-on-solid growth model

#### 3.1. Definition of the model

As a microscopic realization of the KPZ equation let us consider a restricted solid-on-solid (RSOS) growth model on a one-dimensional lattice with  $L$  sites and periodic boundary conditions. The configuration  $C$  of the interface is characterized by height variables  $h_i \in \mathbb{Z}$  attached to the lattice sites  $i$  which obey the restriction

$$|h_i - h_{i \pm 1}| \leq 1. \quad (10)$$

The model evolves random-sequentially by deposition and evaporation of particles with rates  $q_{a,b,c}$  and  $p_{a,b,c}$  according to the rules shown in Fig. 1. Because of (10), it is

convenient to describe the configurations of the interface in terms of charges

$$\sigma_i := h_{i+1} - h_i \in \{0, \pm 1\} \quad (11)$$

which evolve random-sequentially according to the rules

$$\begin{array}{ccccccc}
 -+ & \xrightleftharpoons[p_a]{q_a} & 00 & \xrightleftharpoons[p_b]{q_b} & +- & 0+ & \xrightleftharpoons[p_c]{q_c} & +0 & -0 & \xrightleftharpoons[p_c]{q_c} & 0-
 \end{array} \quad (12)$$

Since deposition and evaporation correspond to negative and positive charge displacements, the average interface velocity  $v = \frac{d}{dt} \langle h_i \rangle$  is given by

$$v = (q_b - p_a) \langle 00 \rangle + q_0 \langle -+ \rangle - p_b \langle +- \rangle + q_c (\langle 0+ \rangle + \langle -0 \rangle) - p_c (\langle +0 \rangle + \langle 0- \rangle), \quad (13)$$

where  $\langle \sigma_i \sigma_{i+1} \rangle$  is the probability to find two specific charges at neighboring sites. Likewise the entropy exported to the environment is given by

$$\begin{aligned}
 \dot{s}_m = & \left( q_b \ln \frac{q_b}{p_b} - p_a \ln \frac{q_a}{p_a} \right) \langle 00 \rangle + q_a \ln \frac{q_a}{p_a} \langle -+ \rangle - p_b \ln \frac{q_b}{p_b} \langle +- \rangle \\
 & + q_c \ln \frac{q_c}{p_c} (\langle 0+ \rangle + \langle -0 \rangle) - p_c \ln \frac{q_c}{p_c} (\langle +0 \rangle + \langle 0- \rangle).
 \end{aligned} \quad (14)$$

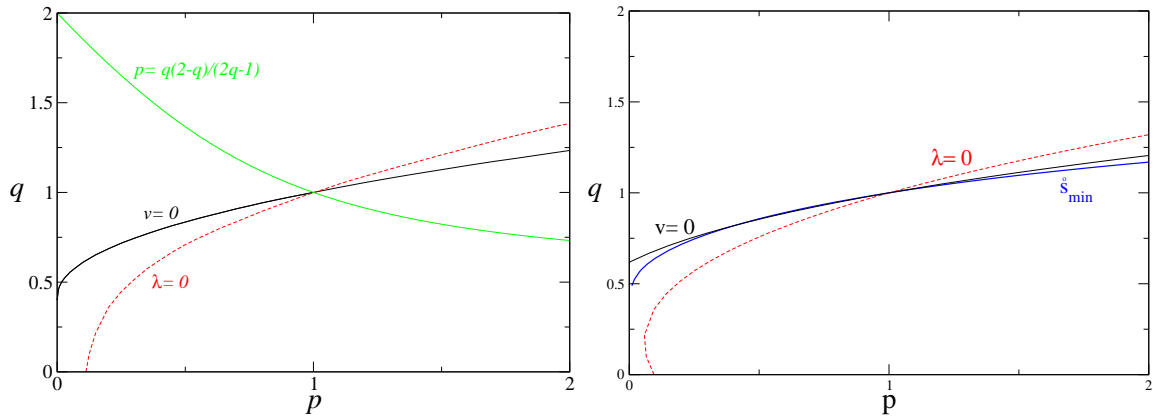
The interface slope  $m$  is defined by

$$m = \frac{1}{L} \sum_{i=1}^L \sigma_i. \quad (15)$$

Since we consider periodic boundary conditions we have  $m = 0$  and, therefore, the number of positive charges is equal to the number of negative charges. One way of introducing a non-zero interface slope in the RSOS model is to change the RSOS restriction between sites  $L$  and  $1$  so that  $h_L - h_1 = H, H \pm 1$ . In this case the interface slope is given by  $m = H/L$ . It can be shown that the  $\lambda$  term in the KPZ equation can be defined for a microscopic model. It is related to how the velocity depends on the interface slope in the following way [13],

$$v(m) = v + (\lambda/2)m^2. \quad (16)$$

We assume that a nonequilibrium stationary state of the RSOS model is characterized by  $v$  and  $\lambda$ , this means that it is characterized by the interface velocity and by how the velocity depends on the interface slope. In the following we calculate the entropy production, the velocity and the parameter  $\lambda$  defined above. We are interested in investigating how the entropy production is a function of these two quantities.



**Figure 2.** Left: Phase diagram of the RSOS model obtained from numerical simulations and the line (18) where the model is exactly solvable. Right: Mean field phase diagram (see text).

### 3.2. Entropy production: Exact results

As shown in Appendix A, the RSOS model can be solved exactly in the thermodynamic limit  $L \rightarrow \infty$  in a certain parameter subspace [25]. To simplify the analysis, we will from now on restrict ourselves to a two-dimensional section of the parameter space [26]

$$q_a = q_b = q_c = q \quad p_a = p \quad p_b = p_c = 1 \quad (17)$$

controlled by two parameters  $q$  and  $p$ . In this section the exactly solvable subspace corresponds to the line

$$p = \frac{2q - q^2}{2q - 1}. \quad (18)$$

As shown in the phase diagram in the left panel of Fig. 2, this line intersects with the lines for  $v = 0$  and  $\lambda = 0$  at the point  $p = q = 1$ , where the model can be shown to obey detailed balance [26]. Note that the lines  $v = 0$  and  $\lambda = 0$  intersecting at the detailed balance point is in agreement with the assumption that  $v$  and  $\lambda$  characterize the nonequilibrium stationary state of the RSOS model.

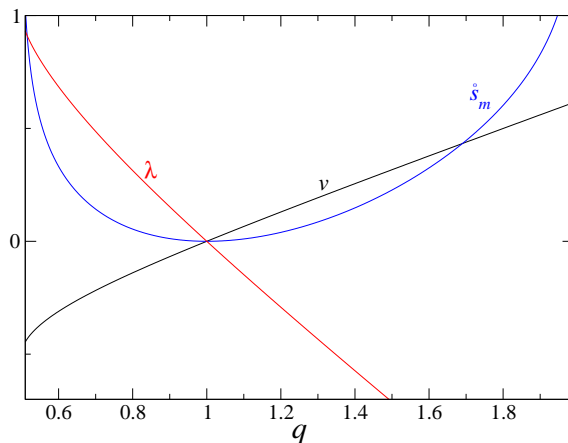
Along the exactly solvable line, the density of charges (cf. Eq. (A.12)) is given by

$$\rho = \frac{2}{2 + \sqrt{(2q - 1)/q}}. \quad (19)$$

As shown in appendix B, the two-point function in the charge basis factorizes in the exactly solvable line, so that the velocity  $v$ , the nonlinearity coefficient  $\lambda$ , and the entropy production rate  $\dot{s}_m$  are given by

$$v = (q - 1)\rho, \quad (20)$$

$$\lambda = -(q - 1) \frac{\rho}{4q(1 - \rho)} \left[ 1 + 6q - 2\rho(2q + 1) \right], \quad (21)$$



**Figure 3.**  $v(q)$ ,  $\lambda(q)$  and  $\dot{s}_m(q)$  along the exactly solvable line.

$$\dot{s}_m = (q - 1) \left( \frac{\rho^2}{2} \ln \frac{2q - 1}{2 - q} + \frac{\rho}{2} (2 - \rho) \ln q \right). \quad (22)$$

In Fig 3 we plot the three quantities as a function of  $q$ . We see that the entropy production is always positive and it grows as the distance from the equilibrium point  $q = 1$  increases. The velocity and  $\lambda$  have opposite signs and as their absolute values increase the entropy production increases. This is in agreement with the idea that the entropy production gives a measure of how far the system is from equilibrium.

### 3.3. Entropy production: Approximate results

The two-point function in the charge basis factorizes only on the line  $p = q(2 - q)/(2q - 1)$ . In order to access the whole phase diagram we now assume that it factorizes for any  $p, q$ , which is analogous to a pair-mean field approach in the height representation [27]. Details of the calculation are presented in Appendix B. Within this mean-field approximation, from (B.5), the density of charges in the stationary state (at zero slope) is given by

$$\rho = 2 / (2 + ((q + p)/(q + 1))^{-1/2}). \quad (23)$$

Repeating the above calculations we find that the interface velocity vanishes for

$$p = -q + q^2 + q^3. \quad (24)$$

and that the coefficient of the non-linear term in the KPZ equation vanishes for

$$p = (-3 - q + 5q^2 + q^3 \pm \sqrt{5 + 10q - 9q^2 - 20q^3 + 3q^4 + 10q^5 + q^6}) / 8, \quad (25)$$

where the sign in front of the square root is positive for  $p > 1$  and negative for  $p < 1$ . As shown in the right panel of Fig. 2, these lines approximate the true lines, cross at the equilibrium point, and have the same type of curvature. In addition, the figure

shows the line where the entropy production obtained from equations (23) and (B.8), is minimal. This line does not coincide with the line  $v = 0$  although it is very close to it. Similar situations have been observed in other models [9–11], where the entropy production was found to peak near the critical point. In the present case, this happens because although the entropy production strongly depends on  $v$ , it also depends on  $\lambda$ .

In principle  $\dot{s}_m$  can be considered functions of  $v$  and  $\lambda$ . However, we could find analytical expressions only for  $v = 0$ , where  $p = -q + q^2 + q^3$ . On this line, from (23) and (B.7) we find

$$\lambda = (q - 1)/q. \quad (26)$$

For large  $q$  we have  $\lambda \rightarrow 1$  and at  $q = (\sqrt{5} - 1)/2$ , which corresponds to  $p = 0$ ,  $\lambda = -(\sqrt{5} - 1)/2 \approx -0.618$ . From (B.8) we have

$$\dot{s}_m = -\frac{\lambda}{(\lambda - 3)^2} \ln \left( \frac{(\lambda - 1)^3}{\lambda^2 - \lambda - 1} \right). \quad (27)$$

where the above relation is valid for  $-(\sqrt{5} - 1)/2 < \lambda < 1$ . As expected, the entropy production is zero at  $\lambda = 0$  and grows as the absolute value of  $\lambda$  grows. Note that it diverges for  $q \rightarrow (\sqrt{5} - 1)/2$  and  $q \rightarrow \infty$ . Again the results are in agreement with the idea that entropy production gives a quantitative measure of the distance from equilibrium.

#### 4. The Gallavotti-Cohen symmetry

The fluctuation relation states that

$$\frac{P(\Delta S_m)}{P(-\Delta S_m)} = \exp(\Delta S_m), \quad (28)$$

where  $P(\Delta S_m)$  denotes probability of the variation of the medium entropy during time  $\Delta T$  and the above relation is valid for  $\Delta T \rightarrow \infty$ . As we show below it leads to the GC symmetry, which is a symmetry in the large deviation function associated with  $P(\Delta S_m)$ . In order to do that in a more general framework we first introduce time-integrated currents and the large deviation function associated with them.

Time-integrated currents are a functional of the stochastic path, which changes only when the underlying process jumps to a different configuration. They are defined by an increment  $\theta_{C \rightarrow C'}$  that depends on the transition rates and is antisymmetric, i.e.  $\theta_{C \rightarrow C'} = -\theta_{C' \rightarrow C}$ . As before we consider the path  $C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_N$  starting at time  $T_0$  and finishing at time  $T$ , where  $N$  is the total number of jumps and  $\Delta T = T - T_0$ . A time-integrated current takes the generic form

$$J = \sum_{k=1}^N \theta_{C_{k-1} \rightarrow C_k}. \quad (29)$$



In the case  $\theta_{C \rightarrow C'} = \ln \frac{w_{C \rightarrow C'}}{w_{C' \rightarrow C}}$ ,  $J$  is just the entropy change  $\Delta S_m$  (4). Another current we are interested in is the variation of height, where  $\theta_{C \rightarrow C'}$  is +1 for a deposition and -1 for an evaporation.

The average of the time integrated current (29) is

$$\langle J \rangle = \int_0^{\Delta T} dt \sum_{C, C'} \theta_{C \rightarrow C'} P(C, t) w_{C \rightarrow C'} \quad (30)$$

In the long time limit, we have by ergodicity

$$\frac{J}{\Delta T} \rightarrow \sum_{C, C'} \theta_{C \rightarrow C'} P(C) w_{C \rightarrow C'}, \quad (31)$$

where  $P(C)$  is the stationary state probability distribution. The large deviation function associated with this time-integrated current is related to deviations of the current from its average value. The large deviation function  $I(j)$  is defined by [28–30]

$$P(j) \approx \exp[-\Delta T I(j)], \quad (32)$$

where  $P(j)$  denotes the probability that  $j = \frac{J}{\Delta T}$  and the above relation is valid for  $\Delta T \rightarrow \infty$ . The GC symmetry is a symmetry of the large deviation function that reads [17]

$$I(j) - I(-j) = -Ej \quad (33)$$

The fluctuation relation for the external medium entropy (28) corresponds to  $E = 1$  and was first proved, for jump processes, by Lebowitz and Spohn [19]. The central idea is to introduce the deformed time evolution operator

$$\tilde{H}_{CC'}(s) = \begin{cases} -w_{C' \rightarrow C} \exp(-s\theta_{C' \rightarrow C}) & \text{if } C \neq C' \\ \sum_{C''} w_{C \rightarrow C''} & \text{if } C = C' \end{cases}. \quad (34)$$

Note that for  $s = 0$  we get the usual time evolution operator for the master equation. Consider the joint probability  $P_J(C, T)$  of being at a state  $C$  at time  $T$  with time-integrated current  $J$ . The Laplace transform of it is

$$\tilde{P}_s(C, T) = \sum_J P_J(C, T) \exp(-sJ). \quad (35)$$

From the master equation (2) we obtain [19]

$$\frac{d}{dT} \tilde{P}_s(C, T) = - \sum_{C'} \tilde{H}_{CC'}(s) \tilde{P}_s(C', T). \quad (36)$$

Since  $\tilde{H}(s)$  is a Perron-Frobenius operator, it is possible to relate the generating function of the time-integrated current  $J$  with its minimum eigenvalue in the following way,

$$\langle \exp(-sJ) \rangle \sim \exp(-\Delta T \hat{I}(s)), \quad (37)$$

where  $\hat{I}(s)$  is the minimum eigenvalue of the matrix  $\tilde{H}(s)$ . The Gärtner-Ellis theorem [28–30] states that the large deviation function (32) is the Legendre transform of  $\hat{I}(s)$ , i.e.,

$$I(j) = \min_s \left( \hat{I}(s) - sj \right). \quad (38)$$

Therefore the GC symmetry (33) is equivalent to

$$\hat{I}(s) = \hat{I}(E - s). \quad (39)$$

In the case where the time integrated current is the external medium entropy (4), we have

$$\tilde{H}_{CC'}(s) = \begin{cases} -w_{C' \rightarrow C}^{1-s} w_{C \rightarrow C'}^s & \text{if } C \neq C' \\ \sum_{C''} w_{C \rightarrow C''} & \text{if } C = C' \end{cases}, \quad (40)$$

which clearly has the symmetry  $\tilde{H}^T(s) = \tilde{H}(1 - s)$ . From this condition, follows (39) with  $E = 1$ .

Let  $w_{C \rightarrow C'}^{eq}$  be some equilibrium transition rates related to an equilibrium distribution  $P^{eq}(C)$ , where detailed balance is satisfied. A sufficient condition for a time-integrated current corresponding to transition rates  $w_{C \rightarrow C'}$  to satisfy the GC symmetry (33), is that a quantity  $E$  exists, such that these rates take the form [23]

$$w_{C \rightarrow C'} = w_{C \rightarrow C'}^{eq} \exp\left(\frac{E}{2} \theta_{C \rightarrow C'}\right). \quad (41)$$

Equivalently to (41), we have

$$w_{C \rightarrow C'} = P_{eq}^{-1}(C) w_{C' \rightarrow C} P_{eq}(C') \exp(E \theta_{C \rightarrow C'}) \quad (42)$$

Defining  $P_{eq}$  as a diagonal matrix with elements  $P_{eq}(C)$ , it then follows, from the last relation,

$$\tilde{H}^T(s) = P_{eq}^{-1} \tilde{H}(E - s) P_{eq}, \quad (43)$$

which directly gives (39). We remark that time-integrated currents, such that an  $E$  satisfying (41) can be found, are proportional to the entropy (up to boundary terms) in the long time limit (see [19, 31]). More precisely,

$$\Delta S_m = \sum_{k=1}^N \ln \frac{P^{eq}(C_k)}{P^{eq}(C_{k-1})} + E \sum_{i=1}^N \theta_{C_{k-1} \rightarrow C_k} = \ln P^{eq}(C_N) - \ln P^{eq}(C_0) + EJ. \quad (44)$$

Therefore, condition (41) is very restrictive in the sense that very few currents satisfy it.

Nevertheless, if one considers the joint probability of more than one time-integrated current a more general relation can be found [19]. Let us take  $k$  currents. Similar to

(41), a sufficient condition for a fluctuation relation including  $k$  currents is that the rates take the form

$$w_{C \rightarrow C'} = P_{eq}^{-1}(C) w_{C' \rightarrow C} P_{eq}(C') \exp \left( \sum_{i=1}^k E_i \theta_{C \rightarrow C'}^i \right). \quad (45)$$

We define the large deviation function associated with the joint probability distribution of the currents  $J_i$  by

$$P(j_1, j_2, \dots, j_k) \approx \exp(-\Delta T I(j_1, j_2, \dots, j_k)). \quad (46)$$

Analogously to (34) we can define the deformed operator

$$\tilde{H}_{CC'}(s_1, s_2, \dots, s_k) = \begin{cases} -w_{C' \rightarrow C} \exp(-\sum_{i=1}^k s_i \theta_{C' \rightarrow C}^i) & \text{if } C \neq C' \\ \sum_{C''} w_{C \rightarrow C''} & \text{if } C = C' \end{cases}, \quad (47)$$

and relation (45) gives

$$\tilde{H}^T(s_1, s_2, \dots, s_k) = P_{eq}^{-1} \tilde{H}(E_1 - s_1, E_2 - s_2, \dots, E_k - s_k) P_{eq}. \quad (48)$$

From the last relation we have a multidimensional version of (39), which reads

$$\hat{I}(s_1, s_2, \dots, s_k) = \hat{I}(E_1 - s_1, E_2 - s_2, \dots, E_k - s_k). \quad (49)$$

In terms of the large deviation function we have the following generalized GC symmetry [19],

$$I(j_1, j_2, \dots, j_k) - I(-j_1, -j_2, \dots, -j_k) = -\sum_{i=1}^k E_i j_i. \quad (50)$$

## 5. Fluctuations of the interface height

Let us consider the RSOS model again. The physically meaningful time-integrated current is the variation of height. Along the line  $p = 1$  the height is just the entropy multiplying  $\ln q$ . Therefore, for  $p = 1$  the probability distribution of the variation of height displays the GC symmetry (33) with  $E = \ln q$ . For  $p \neq 1$  the relation between height and entropy becomes more complicated, and we would like to investigate the symmetry of the large deviation function associated with the probability distribution of height for this case. In other words, we would like to know if a symmetric large deviation function (related to height) is a general property of the RSOS model or if it holds only for specific microscopic rules ( $p = 1$ ). In the rest of this section we show that if a second time-integrated current is considered, fluctuation relations for the joint probability of two time-integrated currents that holds in the whole phase diagram can be found. We then proceed to analyze the symmetry of the Legendre transform of large deviation function associated with height for small systems ( $L = 3, 4$ ).

### 5.1. Joint probability of two time-integrated currents

We introduce the time-integrated currents  $\Delta H_q$  and  $\Delta H_{qp}$  which are defined in the following way. The increment  $\theta_{C \rightarrow C'}^q$  of  $\Delta H_q$  is  $-1$  for an evaporation taking place with rate 1,  $+1$  for the deposition related to the reverse transition, and 0 otherwise. In a same way, the increment  $\theta_{C \rightarrow C'}^{qp}$  of  $\Delta H_{qp}$  is  $-1$  for an evaporation taking place with rate  $p$ ,  $+1$  for the deposition related to the reverse transition and 0 otherwise. Note that, in the present case (45) is verified on the form

$$w_{C \rightarrow C'} = w_{C' \rightarrow C} \exp \left( (\ln q) \theta_{C \rightarrow C'}^q + \left( \ln \frac{q}{p} \right) \theta_{C \rightarrow C'}^{qp} \right). \quad (51)$$

Then, relation (50) becomes

$$I(h_q, h_{qp}) - I(-h_q, -h_{qp}) = -\ln(q)h_q - \ln(q/p)h_{qp} \quad (52)$$

In order to include the height as a current we note that in terms of  $\Delta H_q$  and  $\Delta H_{qp}$ , the height variation (multiplied by the system size  $L$ ),  $\Delta H$ , is given by

$$\Delta H = \Delta H_q + \Delta H_{qp} \quad (53)$$

Hence, (52) implies in

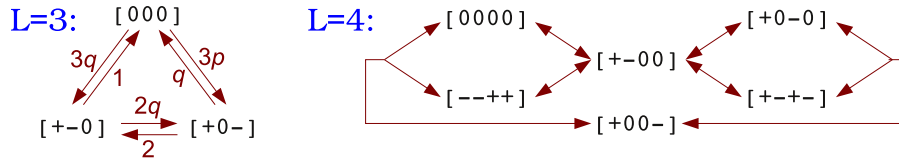
$$\begin{aligned} I(h, h_{qp}) - I(-h, -h_{qp}) &= -\ln(q)h + \ln(p)h_{qp} \\ I(h, h_q) - I(-h, -h_q) &= -\ln(q/p)h + \ln(p)h_q. \end{aligned} \quad (54)$$

The height appears explicitly in these relations and they are valid in the entire phase diagram. Therefore, it is possible to find fluctuation relations for the height if a second time-integrated current is considered. Next, we consider the probability distribution of the variation of height, without a second time-integrated current.

### 5.2. A symmetry different from the GC symmetry

We consider the RSOS model for very small system sizes starting with  $L = 3$ , where the model has 7 states in the charge representation. In order to visualize the network of transitions in this space, let us denote by  $[\sigma_1, \dots, \sigma_L]$  all configurations which differ only by translation, e.g.  $[+ - 0] = \{+ - 0, 0 + -, -0 +\}$ . As shown in the left panel of Fig. 4, the transitions for  $L = 3$  form a simple cycle. Although each transition contributes differently to  $\Delta H$  and  $\Delta S_m$  it is clear from the value of the rates (i.e. Fig. 4) that for each completed cycle the resulting changes will have the fixed ratio  $\frac{\Delta S_m}{\Delta H} = \ln(q) - \frac{1}{3} \ln(p)$ . Therefore, in the long-time limit  $\Delta H$  will be distributed in the same way as  $\Delta S_m$  and thus trivially displays the GC symmetry. More precisely, it is possible to find a similarity transformation of the form (43) with  $E = \ln(q) - \frac{1}{3} \ln(p)$ .

For  $L = 4$  the situation is different. As shown in the right panel of Fig. 4, the transition network is a loop consisting of two smaller cycles. If the system passes the



**Figure 4.** Network of transitions in the charge representation of the RSOS model with  $L = 3$  (left) and  $L = 4$  sites (right).

whole loop it produces both entropy and height, but if it circulates within the small cycles it generates only entropy. This means that entropy and height are no longer coupled and are expected to be distributed differently.

To find out whether the distribution of  $\Delta H$  is still symmetric for  $L = 4$ , let us follow the procedure described in the previous section. In the charge basis the deformed evolution operator  $\tilde{H}(s)$  turns out to be a  $19 \times 19$  matrix of which the lowest eigenvalue has to be found. The problem can be simplified even further by defining the  $6 \times 6$  matrix

$$T_{\kappa\kappa'}(s) := \sum_{c \in \kappa} \tilde{H}_{cc'}(s), \quad (55)$$

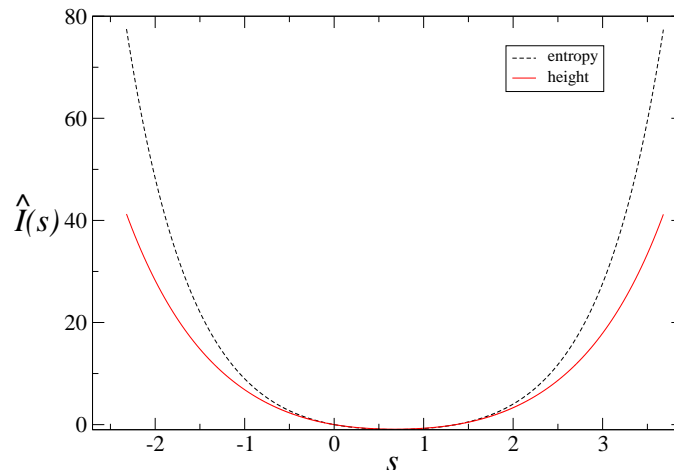
where  $\kappa$  and  $\kappa'$  denote the six sets of charge configurations which differ only by translations and  $C'$  is one of the states represented by  $\kappa'$ . In the basis  $\{[0000], [- - ++], [+ - 00], [+ - + -], [+0 - 0], [+00 -]\}$ , this matrix reads

$$T_{\kappa\kappa'}(s) = \begin{pmatrix} -4p - 4q & 0 & e^s & 0 & 0 & qe^{-s} \\ 0 & -1 - q & pe^s & 0 & 0 & qe^{-s} \\ 4qe^{-s} & qe^{-s} & -1 - p - 3q & 2e^s & 2e^s & 0 \\ 0 & 0 & qe^{-s} & -2 - 2q & 0 & pe^s \\ 0 & 0 & 2qe^{-s} & 0 & -2 - 2q & 2e^s \\ 4pe^s & e^s & 0 & 2qe^{-s} & 2qe^{-s} & -2 - p - 2q \end{pmatrix} \quad (56)$$

Since the eigenvector corresponding to the lowest eigenvalue of  $\tilde{H}(s)$  is translationally invariant, there will be a corresponding eigenvector of  $T(s)$  with the same eigenvalue. Hence  $\hat{I}(s)$  can be computed by determining the lowest eigenvalue of the matrix given above.

Computing its determinant we find that  $\hat{I}(s)$  vanishes for  $s = 0$  and  $E' := s = \frac{1}{4}(\ln 3q^4 - \ln(2p + p^2))$  and that the characteristic polynomial and hence the spectrum is invariant under the replacement  $s \rightarrow E' - s$ . Therefore, the RSOS model with  $L = 4$  sites has a symmetric large deviation function (associated with height). Moreover, the function  $\hat{I}(s)$  for the height current – even when appropriately rescaled – differs from the corresponding function for the entropy (see Fig. 5), showing that entropy and height are not proportional to each other for  $\Delta T \rightarrow \infty$ . In Appendix C we present a different proof for this symmetry, without using the matrix  $\tilde{H}(s)$ .

The GC symmetry has a clear physical interpretation. It is a result of the fact that



**Figure 5.** Numerically determined Legendre transform  $\hat{I}(s)$  of the large deviation function of the entropy (dashed line) rescaled by  $s \rightarrow s / \frac{\ln 3q^4 - \ln(2p+p^2)}{4}$ , and for the height (solid line) in the RSOS model with  $p = 0.1$  and  $q = 2$ .

the exponential of the variation of the entropy for a given stochastic path is the weight of the path divided by the weight of the time-reversed path (see [19, 23]). Clearly, this is not the case of the height in the RSOS model, therefore the symmetry we found for the  $L = 4$  case is different from the GC symmetry. In Appendix C we show that the symmetry is a result of some subtle arrangement of the stochastic trajectories that leads to a variation of height  $\Delta H$ . Hence, we have shown that currents different from the entropy may also have a symmetric large deviation function.

Unfortunately, this observation does not extend to  $L > 4$ , where a small asymmetry in  $\hat{I}(s)$  can be observed (we calculated  $\hat{I}(s)$  numerically, directly from the matrix  $\tilde{H}(s)$ , for  $L = 5, 6, 7$ ). This may be caused by the increasingly complexity of the network of transitions (see Appendix C).

## 6. Conclusions

In the present paper we studied the entropy production and fluctuation relations for the particular solid-on-solid growth model which belongs to the KPZ universality class. On a particular line of the phase diagram, where the model can be solved exactly, we calculated the entropy production, the velocity, and KPZ-parameter  $\lambda$ . The results are compatible with the idea that  $|\lambda|$  and  $|v|$  can be used as a measure for the distance from equilibrium.

Extending these exact results to the entire phase diagram within a mean field approximation, we obtained analytical expressions for the lines  $v = 0$  and  $\lambda = 0$  which are close to the true lines and exhibit the same curvature. Furthermore, we calculated the entropy production as a function of  $\lambda$  along the line  $v = 0$ . It was found that  $\dot{s}_m$  grows with  $|\lambda|$ , again in agreement with the idea that the entropy production gives a

measure of the distance from criticality.

As we saw, along the line  $p = 1$  height is proportional to entropy and therefore its probability distribution displays the GC symmetry. We showed that if a second time-integrated current is considered a symmetry in the joint probability distribution of two time integrated currents valid in the whole phase diagram can be found. Moreover, we explored the symmetry of the Legendre transform of the large deviation function associated with the variation of height for the  $L = 3, 4$  cases. For  $L = 3$  the height turned out to be asymptotically proportional to the entropy, hence it trivially displays a GC symmetry. A different result emerged for  $L = 4$ : entropy and height are not proportional in the long time limit, nevertheless the large deviation function related to height is still symmetric. As we showed this a new symmetry different from the GC symmetry.

We would like to point out that the fluctuation relation for the entropy production is very general, holding for any Markov process with reversible transitions rates. Contrarily, it is valid for very specific currents, that is, currents that become proportional to the entropy in the long time limit. Here we found an example of a current different from the entropy with a symmetric large deviation function. It would be interesting to investigate what currents in Markov processes, different from the entropy, have a symmetric large deviation function and what may be the physical origin of the symmetry.

### Acknowledgements

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## Appendix A. Exact calculation of the stationary state of the RSOS model

Here we provide more details on how the stationary state of the RSOS model expressed in the charge basis can be computed exactly in a certain subspace of the parameters by using the method introduced by Neergaard and den Nijs [25]. Let us assume that the probability to find the stationary system is proportional to

$$f_{M,E} = x^M y^E, \quad (\text{A.1})$$

where  $x \leq 1$  and  $y \leq 1$  are certain numbers depending on the rates and

$$M = \frac{1}{2} \sum_j \sigma_j^2, \quad E = \sum_j j \sigma_j \quad (\text{A.2})$$

are the total number of the charges (divided by two) and their electrostatic energy, respectively. Inserting this ansatz into the master equation yields

$$\begin{aligned}
0 = & n_{+-}[q_b f_{M-1,E+1} - p_b f_{M,E}] + n_{-+}[p_a f_{M-1,E-1} - q_a f_{M,E}] \\
& + n_{00}[q_a f_{M+1,E+1} + p_b f_{M+1,E-1} - (q_b + p_a) f_{M,E}] \\
& + (n_{0-} + n_{+0})[q_c f_{M,E-1} - p_c f_{M,E}] \\
& + (n_{-0} + n_{0+})[p_c f_{M,E+1} - q_c f_{M,E}]
\end{aligned} \tag{A.3}$$

where  $n_{\sigma\sigma'}$  is the probability of finding a pair  $\sigma\sigma'$  of nearest neighbors sites in a configuration with  $M$  charges and electrostatic energy  $E$ . Using the relation

$$n_{\sigma} = \sum_{\sigma'} n_{\sigma\sigma'} = \begin{cases} M/L & \text{if } \sigma = \pm 1 \\ 1 - 2M/L & \text{if } \sigma = 0 \end{cases} \tag{A.4}$$

one obtains the constraint

$$n_{0-} + n_{+0} + 2n_{+-} = n_{-0} + n_{0+} + 2n_{-+}. \tag{A.5}$$

meaning that there are four independent variables in the master equation (A.4), namely  $n_{00}$ ,  $n_{+-}$ ,  $n_{-+}$  and  $(n_{0-} + n_{+0} + 2n_{+-})$ , and that the corresponding coefficients have to vanish. The coefficient of  $(n_{0-} + n_{+0} + 2n_{+-})$  vanishes if

$$(q_c - p_c y^{-1})(1 - y) = 0. \tag{A.6}$$

This equation has two solutions. The first one,  $y = q_c/p_c$ , combined with the other constraints, reduces to the condition of detailed balance (see [25]). The second solution  $y = 1$ , on which we will focus in the following, is much more interesting because it extends to the nonequilibrium regime of the model. In this case the configurational probability distribution becomes independent of the electrostatic energy  $E$ .

For this solution the coefficient of  $n_{00}$  gives

$$x = \frac{q_b + p_a}{q_a + p_b}. \tag{A.7}$$

The two remaining coefficients of  $n_{+-}$  and  $n_{-+}$  lead to the same relation, namely

$$q_c - p_c = \frac{q_b q_a - p_a p_b}{2(q_b + p_a)}. \tag{A.8}$$

Therefore, in the region of the phase space where the constraint (A.8) is satisfied the probability of a configuration with  $M$  charges in system of size  $L$  in the stationary state is exactly given by

$$P(M) = Z_{M,L}^{-1} x^M \tag{A.9}$$

where

$$Z_{M,L} = \sum_{M=0}^{L/2} \frac{L!}{M! M! (L - 2M)!} x^M \tag{A.10}$$



is the partition function and  $x$  is given by (A.7). In finite systems neighboring sites are still correlated because of charge conservation. For example, in a system with 2 sites the only possible configurations are 00, +-, and -+ and thus their probability distribution does not factorize. However, in the limit  $L \rightarrow \infty$  this conservation law will be less and less important, meaning that the state factorizes asymptotically:

$$\langle \sigma \sigma' \rangle = \langle \sigma \rangle \langle \sigma' \rangle + o(1/L), \quad (\text{A.11})$$

Moreover, in the large  $L$  limit the sum in (A.10) is dominated by its maximum and the density of charges  $\rho = 2M/L$  is given by

$$\rho = \frac{2}{2 + x^{-1/2}}. \quad (\text{A.12})$$

## Appendix B. Mean field calculation of $\lambda$ and $\dot{s}_m$

To compute the value of the coefficient  $\lambda$  in the KPZ equation one has to consider a tilted interface in the stationary state and to investigate how the interface velocity  $v$  varies with the slope  $m$ . According to equation (16),  $\lambda$  is then given by

$$\lambda = \frac{1}{m} \frac{d}{dm} v(m) \quad (\text{B.1})$$

where  $v(m)$  is given by Eq. (13).

To compute  $v(m)$  we assume that  $\sigma_i \sigma_{i+1}$  factorizes. Of course this is only correct for  $m = 0$  in the parameter subspace constrained by (A.8). However, it is reasonable to expect that the stationary state still factorizes if  $m$  is sufficiently small. With this assumption we have

$$\langle + \rangle = \frac{\rho(m) + m}{2}, \quad \langle - \rangle = \frac{\rho(m) - m}{2}, \quad \langle 0 \rangle = 1 - \rho(m) \quad (\text{B.2})$$

with some unknown function  $\rho(m)$  for the charge density, so that the interface velocity (13) is given by

$$\begin{aligned} v(m) = & (q_b - p_a)(1 - \rho(m))^2 + \frac{1}{4}(q_a - p_b)(\rho(m)^2 - m^2) \\ & + (q_c - p_c)(1 - \rho(m))\rho(m); \end{aligned} \quad (\text{B.3})$$

To compute  $\lambda$  we need to know the derivative of  $\rho(m)$ . Since

$$\frac{d}{dt} \rho(t) = (q_b + p_a) \langle 00 \rangle - q_a \langle -+ \rangle - p_b \langle +- \rangle \quad (\text{B.4})$$

vanishes in the stationary state, we have

$$(q_b + p_a)(1 - \rho(m))^2 - \frac{1}{4}(q_a + p_b)(\rho(m)^2 - m^2) = 0. \quad (\text{B.5})$$

Differentiating with respect to  $m$  we obtain the expression

$$m^{-1} \frac{d}{dm} \rho(m) = [\rho + 4 \frac{(q_b + p_a)}{q_a + p_b} (1 - \rho)]^{-1}. \quad (\text{B.6})$$

which tends to  $\frac{1}{2x}$  in the limit  $m \rightarrow 0$ . Inserting this result into (B.1) yields

$$\lambda = \frac{x^{-1/2}}{4} [(q_a - p_b)\rho - 4(q_b - p_a)(1 - \rho) + 2(q_c - p_c)(1 - 2\rho)] - \frac{(q_a - p_b)}{2}. \quad (\text{B.7})$$

One can obtain the same expression for  $\lambda$  by differentiating the deterministic part of the KPZ equation for the RSOS model [25]. Similarly, we can compute the entropy production rate (14):

$$\dot{s}_m = (q_a \ln \frac{q_a}{p_a} - p_b \ln \frac{q_b}{p_b}) \frac{\rho^2}{4} + (q_b \ln \frac{q_b}{p_b} - p_a \ln \frac{q_a}{p_a})(1 - \rho)^2 + (q_c - p_c) \ln \frac{q_c}{p_c} \rho(1 - \rho). \quad (\text{B.8})$$

Thus we have obtained the quantities  $v$ ,  $\dot{s}_m$  and  $\lambda$  as functions of the density of charges  $\rho$  in the stationary state, assuming factorization of  $\langle \sigma \sigma' \rangle$ . These results are exact in the parameter subspace constrained by (A.8) in the limit  $L \rightarrow \infty$ , where the density of charges is given by (A.12).

### Appendix C. The special symmetry for $L = 4$

In this Appendix we show that the large deviation function associated with the height variation  $I(h) = -\lim_{\Delta T \rightarrow \infty} \frac{1}{\Delta T} \ln P(h)$ , where  $h = \Delta H / \Delta T$ , has the symmetry

$$I(h) - I(-h) = -h \frac{1}{4} \ln \frac{3q^4}{2p + p^2}, \quad (\text{C.1})$$

for the RSOS model with  $L = 4$ .

We start by considering the embedding Markov chain (discrete time) of the Markovian process. The transition matrix reads

$$T_{C \rightarrow C'} = \frac{w_{C \rightarrow C'}}{\lambda_C} \quad (\text{C.2})$$

where  $\lambda_C = \sum_{C' \neq C} w_{C \rightarrow C'}$ . Within the basis used in (56), the transition matrix takes the explicit form

$$T = \begin{pmatrix} 0 & 0 & \frac{1}{3q+p+1} & 0 & 0 & \frac{q}{2q+p+2} \\ 0 & 0 & \frac{p}{3q+p+1} & 0 & 0 & \frac{q}{2q+p+2} \\ \frac{q}{q+p} & \frac{q}{q+1} & 0 & \frac{1}{q+1} & \frac{1}{q+1} & 0 \\ 0 & 0 & \frac{q}{3q+p+1} & 0 & 0 & \frac{p}{2q+p+2} \\ 0 & 0 & \frac{2q}{3q+p+1} & 0 & 0 & \frac{2}{2q+p+2} \\ \frac{p}{q+p} & \frac{1}{q+1} & 0 & \frac{q}{q+1} & \frac{q}{q+1} & 0 \end{pmatrix} \quad (\text{C.3})$$

In this discrete time case we are interested in the probability of having a variation of height  $\Delta H$  with a fixed number of jumps  $N$ , which we denote by  $P(\Delta H|N)$ . Our aim is to obtain all possible paths that lead to a variation of height  $\Delta H$  within  $N$  jumps. Let us assume that  $\Delta H$  and  $N$  are large, such that only periodic sequence of jumps contribute, while any other sequence of jumps will just generate a boundary term that is irrelevant for large  $\Delta H$  and  $N$ . We also consider an initial state  $[+ - 00]$ , which does not impose any restriction since we assume  $\Delta H$  and  $N$  large.

Among the possible periodic sequence of jumps, there are four cycles that increase the height. They are:

- (i)  $[+ - 00] \rightarrow [+0 - 0] \rightarrow [+00 -] \rightarrow [0000] \rightarrow [+ - 00]$
- (ii)  $[+ - 00] \rightarrow [+ - + -] \rightarrow [+00 -] \rightarrow [0000] \rightarrow [+ - 00]$
- (iii)  $[+ - 00] \rightarrow [+0 - 0] \rightarrow [+00 -] \rightarrow [- - ++] \rightarrow [+ - 00]$
- (iv)  $[+ - 00] \rightarrow [+ - + -] \rightarrow [+00 -] \rightarrow [- - ++] \rightarrow [+ - 00]$

These cycles take 4 jumps each and increase height by 4. Defining  $A = (3q + p + 1)(q + 1)(q + p)(2 + 2q + p)$  and  $B = (3q + p + 1)(q + 1)^2(2 + 2q + p)$ , the probability of cycle 1 is  $M_1 = 2q^4/A$ , of cycle 2 is  $M_2 = q^4/A$ , of cycle 3 is  $M_3 = 2q^4/B$  and of cycle 4 is  $M_4 = q^4/B$ . The reversed cycles decrease height by 4 and they have probabilities  $\widetilde{M}_1 = p^2/A$  for cycles 1,  $\widetilde{M}_2 = 2p/A$  for cycles 2,  $\widetilde{M}_3 = p^2/B$  for cycles 3 and  $\widetilde{M}_4 = 2p/B$  for cycle 4. We denote the number of these cycles in some path  $C = C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_N$  by  $m_i$  and the number of reversed cycles by  $\tilde{m}_i$ , with  $i = 1, 2, 3, 4$ .

Besides these four cycles (and their four reversed ones) there are many other periodic sequence of jumps, however they do not change height. For example, it is possible to jump from  $[+0 - 0]$  to  $[+00 -]$  and then jump back. This sequence takes two jumps and it does not change height. Another example, is the cycle  $[+ - 00] \rightarrow [+0 - 0] \rightarrow [+00 -] \rightarrow [+ - + -] \rightarrow [+ - 00]$ , which takes four jumps and also does not change height. We denote by  $\Delta$  the set of independent periodic sequences of jumps that do not change height. Each element of the set,  $i \in \Delta$ , is characterized by its probability  $P_i$  and the number of jumps it takes  $\eta_i$ . The random number  $n_i$  gives the number of time the sequence  $i \in \Delta$  appear in the path  $\{C\}$ .

The probability of the variation of height  $\Delta H$  with a fixed number of jumps  $N$  can be written in the form

$$P(\Delta H|N) = \sum'_{\{C\}} W[\{C\}], \quad (\text{C.4})$$

where  $W[\{C\}]$  is the probability of the path  $\{C\}$  and the sum  $\sum'_{\{C\}}$  is over all paths with  $N$  jumps and variation of height  $\Delta H$ . By commutativity of the measure of the stochastic path, one can order the terms contributing to the probability of a path such that a given trajectory looks like a sequence of elements of  $\Delta$ , followed by a sequence of cycle 1, 2, 3 or 4 (or their reversed). If we have  $\tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3 + \tilde{m}_4 = j$ , then

$m_1 + m_2 + m_3 + m_4 = \Delta H/4 + j$  and  $\sum_{i \in \Delta} n_i \eta_i = N - \Delta H - 8j$ . Therefore, the sum (C.4) becomes

$$\sum_{j=0}^{(N-\Delta H)/8} \left( \sum_{\{\eta_i\}} \delta \left( \sum_{i \in \Delta} n_i \eta_i - N + \Delta H + 8j \right) \left( \prod_{i \in \Delta} P_i^{n_i} \right) \right) \times \\ (M_1 + M_2 + M_3 + M_4)^{\Delta H/4+j} \left( \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4 \right)^j. \quad (\text{C.5})$$

where the term  $(M_1 + M_2 + M_3 + M_4)^{\Delta H/4+j}$  comes from the sum over all possible  $m_1, m_2, m_3, m_4$  subjected to the constraint  $m_1 + m_2 + m_3 + m_4 = \Delta H/4 + j$  and, similarly, the term  $(\widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4)^j$  comes from the sum over all possible  $\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3, \widetilde{m}_4$  subjected to the constraint  $\widetilde{m}_1 + \widetilde{m}_2 + \widetilde{m}_3 + \widetilde{m}_4 = j$ . Analogously, for  $P(-\Delta H | N)$  we obtain

$$\sum_{j=0}^{(N-\Delta H)/8} \left( \sum_{\{\eta_i\}} \delta \left( \sum_{i \in \Delta} n_i \eta_i - N + \Delta H + 8j \right) \left( \prod_{i \in \Delta} P_i^{n_i} \right) \right) \times \\ (M_1 + M_2 + M_3 + M_4)^j \left( \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4 \right)^{\Delta H/4+j}. \quad (\text{C.6})$$

Therefore, the probability of the variation of height with a fixed number of jumps has the symmetry

$$\frac{P(\Delta H | N)}{P(-\Delta H | N)} = \left( \frac{M_1 + M_2 + M_3 + M_4}{\widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4} \right)^{\Delta H/4}. \quad (\text{C.7})$$

Note that right hand side of the last equation does not depend on the number of jumps  $N$ . By applying Bayes formula, we have

$$\frac{P(h)}{P(-h)} = \frac{\sum_{N=\lceil Th \rceil} P(\Delta H | N) P(N)}{\sum_{N=\lceil Th \rceil} P(-\Delta H | N) P(N)} = \left( \frac{M_1 + M_2 + M_3 + M_4}{\widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4} \right)^{\frac{T}{4}h}, \quad (\text{C.8})$$

where  $h = \Delta H / \Delta T$ ,  $P(h)$  is the probability in the continuous time process with fixed  $\Delta T$  and  $P(N)$  is the probability of having  $N$  jumps.

Finally, since

$$M_1 + M_2 + M_3 + M_4 = 3q^4 \left( \frac{1}{A} + \frac{1}{B} \right) \quad \text{and} \quad \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \widetilde{M}_4 = (p^2 + 2p) \left( \frac{1}{A} + \frac{1}{B} \right), \quad (\text{C.9})$$

from (C.8) we obtain the symmetry (C.1).

An important point in the calculation we performed here is that the chosen initial state  $[+ - 00]$  has the property that any cycle that increases height has to go through it (we could also choose  $[+00-]$ ). Without this property it would not be possible to write down equations (C.5) and (C.6), from which we derive the symmetry. As we mentioned, the symmetry is lost for  $L = 5$  where the large deviation function displays a small asymmetry. By drawing the network of states for the  $L = 5$  case, which has 10 states, we observed that there is no state such that a cycle that increases height has to go through it.

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